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DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS

THE EFFECT OF A MOVING LOAD ON A VISCOELASTIC HALF-SPACE

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Office of Naval Research
Project NR 064-417
Contract No. Nonr 266(34)

Technical Report No. 12
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ABSTRACT

Cole and Huth [1] have studied the effect of a line load moving with constant velocity V along the surface of an elastic half-space. The present paper treats the equivalent problem for a viscoelastic (standard solid) material when the velocity V is less than the velocity c_s of shear waves of high frequencies $(\Omega \to \infty)$.

In Reference [1] no solution could be obtained when the velocity V was equal to the velocity of Rayleigh waves. The present analysis yields a solution for this special velocity. It also permits an evaluation of the effects of viscosity at other values of V. In certain ranges these effects are minor, but in other ranges major differences occur due to focusing phenomena.

TABLE OF CONTENTS

PARTIAL LIST OF SYMBOLS

c_s, c_s^*	Velocities of shear waves in the limits $\Omega \to \infty$ and $\Omega \to 0$, respectively
c_p, c_p^*	Velocities of dilatational waves in the limits $\Omega \to \infty$ and $\Omega \to 0$, respectively
$\mathrm{Erf}(x) = \int_0^x e^{-t^2} dt$	, the error function
$\mathrm{Erfc}(x) = \int_x^\infty e^{-t^2} dt$	, the complementary error function
k	Bulk modulus
$K = \frac{c_s^2}{c_p^2}$	Ratio of wave speeds in the limit $\Omega \to \infty$
$M_L = \frac{V}{c_p}$	Dilational Mach number in the limit $\Omega \to \infty$
$M_T = \frac{V}{c_s}$	Shear Mach number in the limit $\Omega \to \infty$
m	Ratio of relaxed to unrelaxed shear modulus
$n = 1 - \frac{4}{3} K(1 - m)$	
P	Load per unit length
T	Relaxation time
t	Time
u_i, u, w	Components of displacement
V	Velocity of the moving load

PARTIAL LIST OF SYMBOLS (CONT'D)

$\bar{x}, \bar{y}, \bar{z}$	Fixed coordinates
x, y, z	Moving coordinates
$\eta = \frac{z}{VT}$	Dimensionless moving coordinate
μ	Shear modulus in the limit $\Omega \to \infty$
$\xi = \frac{x}{VT}$	Dimensionless moving coordinate
ρ	Density
σ_{ij}	Stress components
Φ	Irrotational potential function
Ψ_i , Ψ	Components of equivoluminal vector potential function
ω	Transform parameter

THE EFFECT OF A MOVING LOAD ON A VISCOELASTIC HALF-SPACE

I. Introduction

The response of an elastic half-space due to loads moving on the surface has been considered by Sneddon [1, 2, 3] who obtained formal solutions for general loads, and closed solutions for certain specific cases. Subsequently, Cole and Huth [4] obtained by other methods closed form solutions for the plane problem of a line load progressing with constant velocity on the surface of the half-space. Miles [5] has considered the case of loads with axially symmetric distributions acting over a circular area, the radius of which expands with velocity V, Figure 1. He has demonstrated that the two-dimensional Cole and Huth problem, with an exception, is identical with an asymptotic solution of the three dimensional problem valid in the vicinity of the load front [5]. This confirms the intuitive conclusion that plane steady-state solutions, which are relatively easy to obtain, can be used (with limitations) as approximations in three dimensional situations such as that shown in Figure 1.

The results of [4] indicate that the character of the solution depends on the relative values of the velocity V and of the velocities of dilatational and shear waves in the medium, c_p and c_s, respectively. The three resulting cases were designated in [4] as subsonic, trans-sonic and supersonic. Subsequently, it was noted in [5] that the steady-state solution of [4] breaks down if $V = c_R$, the velocity of Rayleigh waves, such that in this exceptional case the three dimensional problem cannot be approximated by the plane solution.

Lorsch and Freudenthal [6] have considered a related problem, the quasi-static problem of a line load moving with constant velocity on the surface of a viscoelastic (Maxwell solid) half-space.

The plane steady-state problem of a line load progressing with constant velocity V on the surface of a viscoelastic (standard solid) half-space has been treated by Sackman [7], but only for the supersonic case, i.e., when V is larger than all wave velocities in the medium. In the present paper, an alternative case will be considered where V is smaller than the limiting velocity c_s of waves of high ($\lim \Omega \to \infty$) frequencies.

In a standard solid the wave velocities decrease with decreasing frequency Ω, the limiting velocities for $\Omega \to 0$ being $c_p^* < c_p$, $c_s^* < c_s$. The present case will therefore include subcases depending on the value of V relative to c_s^* and c_p^*. The solutions obtained permit an insight into the deviations between the response in elastic and viscoelastic situations in general; further, a solution is presented for the special case, $V = c_R$, where no steady state solution for the elastic half-space exists.

A formal solution of the problem will be obtained by integral transform methods. Due to the complexities introduced by the viscoelastic behavior, closed solutions, do, in general, not exist, and recourse will be taken to asymptotic methods of evaluation. The quantities of interest, the stresses and accelerations, will be obtained. Additional quantities, such as velocities or displacements, could be obtained if desired, except for a free and undeterminable constant in the displacements. The indeterminacy of the displacements is not unexpected; it already exists in the two-dimensional elastic problem [4], and even in the equivalent static one [8]. (The matter is discussed in [5].)

II. Formulation of the Problem

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ designate a stationary coordinate system in the half-space $\bar{z} \geq 0$, Fig. 2. A line load P of unit intensity, positive if downward, moving in the negative $\bar{x}$ - direction with uniform velocity V, may be described by

$$P(\bar{x}, t) = \delta(\bar{x} + Vt) \tag{1}$$

where the symbol δ indicates Dirac's function. The equations of motion are

$$\sigma_{ij,j} = \rho \ddot{u}_i \quad . \tag{2}$$

The stress-displacement relation for a homogeneous isotropic material, elastic in bulk, and viscoelastic in shear is

$$\sigma_{ij} = (k - \frac{2}{3}\tilde{\mu})u_{\ell,\ell}\,\delta_{ij} + \tilde{\mu}(u_{i,j} + u_{j,i}) \tag{3}$$

where k is the elastic bulk modulus. For the standard solid $\tilde{\mu}$ is the operator

$$\tilde{\mu} = \mu \frac{m + T\frac{\partial}{\partial t}}{1 + T\frac{\partial}{\partial t}} \quad . \tag{4}$$

In this relation μ is the "unrelaxed" and $m\mu$ the "relaxed" shear modulus, while T is the relaxation time, (See Ref. 9).

The displacements may be written according to the Helmholtz resolution,

$$u_i = \Phi_{,i} + \epsilon_{ij\ell}\,\Psi_{\ell,j} \tag{5}$$

as functions of the potentials Φ and Ψ_i. In the present problem, the material is in a state of plane strain, $u_2 = 0$, and Eq. (5) reduces to

$$\begin{aligned} u_1 &\equiv u = \Phi,_{\bar{x}} - \Psi,_{\bar{z}} \\ u_3 &\equiv w = \Phi,_{\bar{z}} + \Psi,_{\bar{x}} \end{aligned} \tag{6}$$

where $\Phi = \Phi(\bar{x}, \bar{z})$, $\Psi_1 = \Psi_3 = 0$, and $\Psi_2 \equiv \Psi(\bar{x}, \bar{z})$.

Using Eqs. (3), (4) and (6), the equation of motion (2) becomes

$$\begin{aligned} (k + \tfrac{4}{3}\tilde{\mu})(\Phi,_{\bar{x}\bar{x}} + \Phi,_{\bar{z}\bar{z}}) &= \rho\ddot{\Phi} \\ \tilde{\mu}\,(\Psi,_{\bar{x}\bar{x}} + \Psi,_{\bar{z}\bar{z}}) &= \rho\ddot{\Psi}\ . \end{aligned} \tag{7}$$

The purpose of the paper being the determination of a steady state solution, consider a second coordinate system, Fig. 2, moving with the load, and related to the stationary coordinate system by the Galilean transformation:

$$\begin{aligned} x &= \bar{x} + Vt \\ y &= \bar{y} \\ z &= \bar{z} \\ t &\equiv t \quad . \end{aligned} \tag{8}$$

The steady state solution must satisfy the partial differential equations in x and z,

$$\begin{aligned} \left(k + \frac{4}{3}\,\mu\,\frac{m + VT\frac{\partial}{\partial x}}{1 + VT\frac{\partial}{\partial x}}\right)(\Phi,_{xx} + \Phi,_{zz}) &= \rho V^2\Phi,_{xx} \\ \mu\left(\frac{m + VT\frac{\partial}{\partial x}}{1 + VT\frac{\partial}{\partial x}}\right)(\Psi,_{xx} + \Psi,_{zz}) &= \rho V^2\Psi,_{xx}\ , \end{aligned} \tag{9}$$

with boundary conditions

$$\sigma_{zz}(x,0) \equiv \left[\left(\rho V^2 - 2\mu \frac{m+VT\frac{\partial}{\partial x}}{1+VT\frac{\partial}{\partial x}}\right)\Phi_{,xx} + 2\mu\left(\frac{m+VT\frac{\partial}{\partial x}}{1+VT\frac{\partial}{\partial x}}\right)\Psi_{,xz}\right]_{z=0} = -\delta(x)$$

$$\sigma_{xz}(x,0) \equiv \left[\mu\left(\frac{m+VT\frac{\partial}{\partial x}}{1+VT\frac{\partial}{\partial x}}\right)\left\{2\Phi_{,xz} + \left(2 - \frac{\rho V^2}{\mu}\,\frac{1+VT\frac{\partial}{\partial x}}{m+VT\frac{\partial}{\partial x}}\right)\Psi_{,xx}\right\}\right]_{z=0} = 0\,, \tag{10}$$

and the provision that

$$\sigma_{ij} \to 0 \quad \text{as} \quad z \to \infty \quad . \tag{11}$$

The problem to be solved is now specified by Eqs.(9), (10) and (11).

III. Formal Solution

The solution of Eqs. (9), (10) and (11) may be obtained formally by the use of an integral transform technique. The steady state disturbance from loads moving with a velocity $V < c_s$ is expected to extend to infinity in both directions such that the Fourier transform is appropriate.

The Fourier integral relations

$$\overline{F}(\omega) = \int_{-\infty}^{\infty} F(\xi) e^{i\omega\xi} \, d\xi \tag{12}$$

$$F(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F}(\omega) e^{-i\omega\xi} \, d\omega \tag{13}$$

apply provided $F(x)$ satisfies certain conditions [10, 11].

As a first step the Fourier transforms for the stresses and accelerations will be obtained.

Introducing dimensionless coordinates $\xi = \frac{x}{VT}$, $\eta = \frac{z}{VT}$, Eqs. (9) give:

$$\begin{aligned} &\overline{\Phi}_{,\eta\eta} - \omega^2\left(1 - M_L^2 \frac{1-i\omega}{n-i\omega}\right)\overline{\Phi} = 0 \\ &\overline{\Psi}_{,\eta\eta} - \omega^2\left(1 - M_T^2 \frac{1-i\omega}{m-i\omega}\right)\overline{\Psi} = 0 \end{aligned} \tag{14}$$

while Eqs. (10) give:

$$\begin{aligned} &-\omega^2\left(M_T^2 - 2\,\frac{m-i\omega}{1-i\omega}\right)\Phi(\omega, 0) - 2i\omega\left(\frac{m-i\omega}{1-i\omega}\right)\Psi_{,\eta}(\omega, 0) = -\frac{VT}{\mu} \\ &-2i\omega\,\Phi_{,\eta}(\omega, 0) - \omega^2\left(2 - M_T^2\,\frac{1-i\omega}{m-i\omega}\right)\overline{\Psi}(\omega, 0) = 0 \end{aligned} \tag{15}$$

where the following abbreviations are used: $M_L^2 = \frac{V^2}{c_p^2}$, $M_T^2 = \frac{V^2}{c_p^2}$, $c_p^2 = \frac{k + 4/3\mu}{\rho} = \frac{1}{K} c_s^2$, $c_s^2 = \mu/\rho$, and $n = 1 - 4/3K(1 - m)$. The symbols c_p and c_s indicate the velocities of irrotational and equivoluminal waves, respectively, in the limit of infinite frequency, $\Omega \to \infty$. In addition, Eq. (11) requires that

$$\overline{\Phi}, \overline{\Psi} \text{ are to remain finite as } \eta \to \infty . \tag{16}$$

The solutions of Eqs. (14) are

$$\begin{aligned} \overline{\Phi} &= A_1 e^{\eta\omega\sqrt{1- M_L^2 \frac{1-i\omega}{n-i\omega}}} + A_2 e^{-\eta\omega\sqrt{1- M_L^2 \frac{1-i\omega}{n-i\omega}}} \\ \overline{\Psi} &= B_1 e^{\eta\omega\sqrt{1- M_T^2 \frac{1-i\omega}{m-i\omega}}} + B_2 e^{-\eta\omega\sqrt{1- M_T^2 \frac{1-i\omega}{m-i\omega}}} . \end{aligned} \tag{17}$$

To make the quantities $\omega\sqrt{1- M_L^2 \frac{1-i\omega}{n-i\omega}}$ and $\omega\sqrt{1- M_T^2 \frac{1-i\omega}{m-i\omega}}$ single valued, branch cuts and argument restrictions are made below. Further analysis requires separate consideration of three cases, depending on the value of the velocity V. Denoting the velocities of irrotational and equivoluminal waves in the limit $\Omega \to 0$ by $c_p^* = \sqrt{n} c_p$ and $c_s^* = \sqrt{m} c_s$, the following three possibilities must be considered separately.

Case a: $V < c_s^*$

Case b: $c_s^* < V < c_p^*$

Case c: $c_p^* < V < c_s$

Case a

If the branch cuts shown in Fig. 3 are used with the argument restrictions

$$
\begin{aligned}
&- \pi \leqq \arg\left(1- M_T^2 \frac{1-i\omega}{m-i\omega}\right) < \pi && \text{(a)} \\
&- \pi \leqq \arg\left(1- M_L^2 \frac{1-i\omega}{n-i\omega}\right) < \pi && \text{(b)}
\end{aligned}
\tag{18}
$$

then

$$
\begin{aligned}
\overline{\Phi} &= - \frac{VT}{\mu\omega^2\overline{\Delta}\overline{Q}} (2 - \overline{M}_T^2) e^{\pm \eta\omega\overline{\beta}_L} \\
\overline{\Psi} &= \pm \frac{VT}{\mu\omega^2\overline{\Delta}\overline{Q}} 2i\overline{\beta}_L e^{\pm \eta\omega\overline{\beta}_T}
\end{aligned}
\tag{19}
$$

where the upper signs are to be used for $R(\omega) < 0$, the lower signs for $R(\omega) > 0$, and the following abbreviations are employed:

$$
\begin{aligned}
\overline{Q} &= \frac{m-i\omega}{1-i\omega} \\
\overline{\beta}_L^2 &= 1- \overline{M}_L^2 = 1- M_L^2 \frac{1-i\omega}{n-i\omega} \\
\overline{\beta}_T^2 &= 1- \overline{M}_T^2 = 1- M_T^2 \frac{1-i\omega}{m-i\omega} \\
\overline{\Delta} &= (2 - \overline{M}_T^2)^2 - 4\overline{\beta}_L\overline{\beta}_T \quad .
\end{aligned}
\tag{20}
$$

The exponents thus chosen have negative real parts in the range of ω where they are to be used.

Substitution of Eq. (5) into Eq. (3) makes it evident that the stresses may be written

$$\sigma_{ij} = \sigma_{ij_\Phi} + \sigma_{ij_\Psi} \tag{21}$$

where σ_{ij_Φ} is the portion derived from Φ and σ_{ij_Ψ} the portion derived from Ψ. The accelerations may be written in a similar fashion.

The transforms of the stresses are then

$$\bar{\sigma}_{xx_\Phi} = \frac{(2 + \bar{M}_T^2 - 2\bar{M}_L^2)(2 - \bar{M}_T^2)}{VT\bar{\Delta}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{xx_\Psi} = - \frac{4\bar{\beta}_L\bar{\beta}_T}{VT\bar{\Delta}} e^{\pm \eta\omega\bar{\beta}_T}$$

$$\bar{\sigma}_{yy_\Phi} = \frac{(\bar{M}_T^2 - 2\bar{M}_L^2)(2 - \bar{M}_T^2)}{VT\bar{\Delta}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{yy_\Psi} \equiv 0 \tag{22}$$

$$\bar{\sigma}_{zz_\Phi} = - \frac{(2 - \bar{M}_T^2)^2}{VT\bar{\Delta}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{zz_\Psi} = - \bar{\sigma}_{xx_\Psi}$$

$$\bar{\sigma}_{xz_\Phi} = \pm \frac{2i\bar{\beta}_L(2 - \bar{M}_T^2)}{VT\bar{\Delta}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{xz_\Psi} = \mp \frac{2i\bar{\beta}_L(2 - \bar{M}_T^2)}{VT\bar{\Delta}} e^{\pm \eta\omega\bar{\beta}_T} .$$

The transforms of the accelerations are

$$\bar{\ddot{u}}_\Phi = - \frac{i\omega(2 - \bar{M}_T^2)}{\mu T^2 \bar{\Delta}\ \bar{Q}}\, e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\ddot{u}}_\Psi = \frac{2i\omega\bar{\beta}_L\bar{\beta}_T}{\mu T^2 \bar{\Delta}\ \bar{Q}}\, e^{\pm \eta\omega\bar{\beta}_T}$$

$$\bar{\ddot{w}}_\Phi = \pm \frac{\omega\bar{\beta}_L(2 - \bar{M}_T^2)}{\mu T^2 \bar{\Delta}\ \bar{Q}}\, e^{\pm \eta\omega\bar{\beta}_L} \tag{23}$$

$$\bar{\ddot{w}}_\Psi = \mp \frac{2\omega\bar{\beta}_L}{\mu T^2 \bar{\Delta}\ \bar{Q}}\, e^{\pm \eta\omega\bar{\beta}_T} \quad .$$

Case b

If the branch cuts shown in Fig. 4 are used with the argument restrictions (18b) and

$$0 \leqq \arg\left(1 - M_T^2\, \frac{1-i\omega}{m-i\omega}\right) < 2\pi \tag{24}$$

then

$$\bar{\Phi} = - \frac{VT}{\mu\omega^2\bar{Q}}\ \frac{(2 - \bar{M}_T^2)\, e^{\pm \eta\omega\bar{\beta}_L}}{(2 - \bar{M}_T^2)^2 \mp 4\bar{\beta}_L\bar{\beta}_T}$$

$$\bar{\Psi} = \pm \frac{VT}{\mu\omega^2\bar{Q}}\ \frac{2i\bar{\beta}_L\, e^{\eta\omega\bar{\beta}_T}}{(2 - \bar{M}_T^2)^2 \mp 4\bar{\beta}_L\bar{\beta}_T} \quad . \tag{25}$$

Again, the upper signs apply for $R(\omega) < 0$, the lower signs for $R(\omega) > 0$ and the exponents have negative real parts.

Using the definition (21) and

$$\bar{\Delta}_\mp = (2 - \bar{M}_T^2)^2 \mp 4\bar{\beta}_L\bar{\beta}_T \tag{26}$$

the transforms of the stresses are

$$\bar{\sigma}_{xx_\Phi} = \frac{(2 + \bar{M}_T^2 - 2\bar{M}_L^2)(2 - \bar{M}_T^2)}{VT\bar{\Delta}_{\mp}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{xx_\Psi} = \mp \frac{4\bar{\beta}_L\bar{\beta}_T}{VT\bar{\Delta}_{\mp}} e^{\eta\omega\bar{\beta}_T}$$

$$\bar{\sigma}_{YY_\phi} = \frac{(\bar{M}_T^2 - 2\bar{M}_L^2)(2 - \bar{M}_T^2)}{VT\bar{\Delta}_{\mp}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{YY_\Psi} \equiv 0 \tag{27}$$

$$\bar{\sigma}_{zz_\phi} = - \frac{(2 - \bar{M}_T^2)}{VT\bar{\Delta}_{\mp}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{zz_\Psi} = - \bar{\sigma}_{xx_\Psi}$$

$$\bar{\sigma}_{xz_\phi} = \pm \frac{2i\bar{\beta}_L(2 - \bar{M}_T^2)}{VT\bar{\Delta}_{\mp}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{xz_\Psi} = \mp \frac{2i\bar{\beta}_L(2 - \bar{M}_T^2)}{VT\bar{\Delta}_{\mp}} e^{\eta\omega\bar{\beta}_T} .$$

The transforms of the accelerations are

$$\bar{\ddot{u}}_\Phi = -\frac{i\omega(2 - \bar{M}_T^2)}{\mu T^2 \bar{\Delta}_\mp \bar{Q}} e^{\pm \eta\omega\bar{\beta}_L}$$

$$\bar{\ddot{u}}_\Psi = \pm \frac{2i\omega\bar{\beta}_L\bar{\beta}_T}{\mu T^2 \bar{\Delta}_\mp \bar{Q}} e^{\eta\omega\bar{\beta}_T}$$

$$\bar{\ddot{w}}_\Phi = \pm \frac{\omega\bar{\beta}_L(2 - \bar{M}_T^2)}{\mu T^2 \bar{\Delta}_\mp \bar{Q}} e^{\pm \eta\omega\bar{\beta}_L} \tag{28}$$

$$\bar{\ddot{w}}_\Psi = \mp \frac{2\omega\bar{\beta}_L}{\mu T^2 \bar{\Delta}_\mp \bar{Q}} e^{\eta\omega\bar{\beta}_T} .$$

Case c

If the branch cuts shown in Fig. 5 are used with the argument restriction (24) and a similar restriction for $\bar{\beta}_L^2$ then

$$\bar{\Phi} = -\frac{VT(2 - \bar{M}_T^2)}{\mu\omega^2 \bar{\Delta}\, \bar{Q}} e^{\eta\omega\bar{\beta}_L}$$

$$\bar{\Psi} = \frac{VT2i\bar{\beta}_L}{\mu\omega^2 \bar{\Delta}\, \bar{Q}} e^{\eta\omega\bar{\beta}_T} \tag{29}$$

where the real parts of the exponents are again negative.

The transforms of the stresses are

$$\bar{\sigma}_{xx_\Phi} = \frac{(2 + \bar{M}_T^2 - 2\bar{M}_L^2)(2 - \bar{M}_T^2)}{VT\,\bar{\Delta}}\, e^{\eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{xx_\Psi} = -\frac{4\bar{\beta}_L\bar{\beta}_T}{VT\bar{\Delta}}\, e^{\eta\omega\bar{\beta}_T}$$

$$\bar{\sigma}_{yy_\Phi} = \frac{(\bar{M}_T^2 - 2\bar{M}_L^2)(2 - \bar{M}_T^2)}{VT\,\bar{\Delta}}\, e^{\eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{yy_\Psi} \equiv 0$$

$$\bar{\sigma}_{zz_\Phi} = -\frac{(2 - \bar{M}_T^2)^2}{VT\,\bar{\Delta}}\, e^{\eta\omega\bar{\beta}_L} \tag{30}$$

$$\bar{\sigma}_{zz_\Psi} = -\bar{\sigma}_{xx_\Psi}$$

$$\bar{\sigma}_{xz_\Phi} = \frac{2i\bar{\beta}_L(2 - \bar{M}_T^2)}{VT\,\bar{\Delta}}\, e^{\eta\omega\bar{\beta}_L}$$

$$\bar{\sigma}_{xz_\Psi} = -\frac{2i\bar{\beta}_L(2 - \bar{M}_T^2)}{VT\,\bar{\Delta}}\, e^{\eta\omega\bar{\beta}_T} \quad .$$

The transforms of the accelerations are

$$\bar{\ddot{u}}_\Phi = - \frac{i\omega(2 - \bar{M}_T^2)}{\mu T^2 \bar{\Delta}\ \bar{Q}} e^{\eta\omega\bar{\beta}_L}$$

$$\bar{\ddot{u}}_\Psi = \frac{2i\omega\bar{\beta}_L\bar{\beta}_T}{\mu T^2 \bar{\Delta}\ \bar{Q}} e^{\eta\omega\bar{\beta}_T}$$

$$\bar{\ddot{w}}_\Phi = \frac{\omega\bar{\beta}_L(2 - \bar{M}_T^2)}{\mu T^2 \bar{\Delta}\ \bar{Q}} e^{\eta\omega\bar{\beta}_L} \tag{31}$$

$$\bar{\ddot{w}}_\Psi = - \frac{2\omega\bar{\beta}_L}{\mu T^2 \bar{\Delta}\ \bar{Q}} e^{\eta\omega\bar{\beta}_T} .$$

IV. Asymptotic Approximations for Large Values of η

According to Eqs. (13), (19), (25) and (29), the stresses and accelerations are, in general, given formally by expressions of the form

$$I = \int_{-\infty}^{0} g_-(\omega) e^{\eta\omega h_-(\omega)} d\omega + \int_{0}^{\infty} g_+(\omega) e^{\eta\omega h_+(\omega)} d\omega . \qquad (32)$$

Since the exponents have been chosen with negative real parts and the functions $g(\omega)$ are finite along the real axis with the exception of the point $|\omega| = \infty$ when $V = c_R$, and of the point $\omega = 0$ when V assumes the special value c_R^* discussed following Eqs. (39), the integrals of Eq. (32) are of a class for which asymptotic approximations may be obtained for either large $(\eta \to \infty)$ or small $(\eta \to 0, \xi \to 0)$ values of η: As a result of the integrability of the functions $g(\omega)$ the major contribution to the integrals (32) for large values of η comes from the vicinity of the origin, and Laplace's method as demonstrated in Ref. [12] is applicable.

Case a

In this case the contributions from both potentials are of the general form (32), $I = I_1 + I_2$. Consider the typical contribution from the equivoluminal potential Ψ. In this case

$$h_-(\omega) = \overline{\beta}_T - i\frac{\xi}{\eta} = (\beta_T^* - i\frac{\xi}{\eta}) - \frac{M_T^{*2}(1 - m)}{2m\beta_T^*} i\omega + O(\omega^2) \qquad (33)$$

and

$$h_+(\omega) = -\bar{\beta}_T - i\,\frac{\xi}{\eta} = -(\beta_T^* - i\,\frac{\xi}{\eta}) + \frac{M_T^{*2}(1 - m)}{2m\beta_T^*}\, i\omega + O(\omega^2). \quad (34)$$

Following the procedure outlined in [12] using only the first term of the expressions (33) and (34) one finds for $V \neq c_R^*$

$$I_1 = \int_{-\infty}^{0} g_-(\omega) e^{\eta\omega h_-(\omega)} d\omega \sim \sum_{n=0}^{N} \frac{(-1)^n n! A_n}{(\eta\beta_T^* - i\xi)^{n+1}} \quad (35)$$

where

$$g_-(\omega) e^{\eta\omega[h_-(\omega) - h_-(0)]} = \sum_{n=0}^{\infty} A_n \omega^n . \quad (36)$$

Similarly,

$$I_2 = \int_{0}^{\infty} g_+(\omega) e^{\eta\omega h_+(\omega)} d\omega \sim \sum_{n=0}^{N} \frac{n! B_n}{(\eta\beta_T^* + i\xi)^{n+1}} \quad (37)$$

where

$$g_+(\omega) e^{\eta\omega[h_+(\omega) - h_+(0)]} = \sum_{n=0}^{\infty} B_n \omega^n . \quad (38)$$

Substitution of the appropriate expansion coefficients A_n and B_n into Eqs. (35) and (37) leads to expressions for the stresses and accelerations. Retaining only the leading non-vanishing coefficient, A_n or B_n, respectively, one obtains for $V \neq c_R^*$

$$\sigma_{xx} \sim \frac{\beta_L^*}{\pi\Delta^*}\left[(2+M_T^{*2} - 2M_L^{*2})(2 - M_T^{*2}) \frac{z}{x^2 + \beta_L^{*2}z^2} - 4\beta_T^{*2} \frac{z}{x^2 + \beta_T^{*2}z^2}\right]$$

$$\sigma_{yy} \sim \frac{\beta_L^*}{\pi\Delta^*} (M_T^{*2} - 2M_L^{*2})(2 - M_T^{*2}) \frac{z}{x^2 + \beta_L^{*2}z^2}$$

$$\sigma_{zz} \sim -\frac{\beta_L^*}{\pi\Delta^*}\left[(2 - M_T^{*2})^2 \frac{z}{x^2 + \beta_L^{*2}z^2} - 4\beta_T^{*2} \frac{z}{x^2 + \beta_T^{*2}z^2}\right]$$

$$\sigma_{xz} \sim -\frac{2\beta_L^*(2 - M_T^{*2})}{\pi\Delta^*}\left[\frac{x}{x^2 + \beta_L^{*2}z^2} - \frac{x}{x^2 + \beta_T^{*2}z^2}\right] \tag{39}$$

$$\ddot{u} \sim -\frac{2V^2\beta_L^*}{\pi m\mu\Delta^*}\left[(2 - M_T^{*2}) \frac{xz}{(x^2 + \beta_L^{*2}z^2)^2} - 2\beta_T^{*2} \frac{xz}{(x^2 + \beta_T^{*2}z^2)^2}\right]$$

$$\ddot{w} \sim \frac{V^2\beta_L^*}{\pi m\mu\Delta^*}\left[(2 - M_T^{*2}) \frac{(x^2 - \beta_L^{*2}z^2)}{(x^2 + \beta_L^{*2}z^2)^2} - 2 \frac{(x^2 - \beta_T^{*2}z^2)}{(x^2 + \beta_T^{*2}z^2)^2}\right]$$

where

$$M_L^{*2} = \frac{M_L^2}{n}, \quad M_T^{*2} = \frac{M_T^2}{m}, \quad \beta_L^{*2} = 1 - M_L^{*2}, \quad \beta_T^{*2} = 1 - M_T^{*2},$$

and

$$\Delta^* = (2 - M_T^{*2})^2 - 4\beta_L^*\beta_T^* .$$

The velocity c_R^* defined by $\Delta^* = 0$ corresponds to the Rayleigh velocity for an elastic material having the properties of the relaxed viscoelastic material. When $V = c_R^*$ the expansions (36) and (38) must be replaced by

$$g_-(\omega)e^{\eta\omega[h_-(\omega)-h_-(0)]} = \frac{A_{-1}}{\omega} + \sum_{n=0}^{\infty} A_n\omega^n \tag{40}$$

and

$$g_+(\omega)e^{\eta\omega[h_+(\omega)-h_+(0)]} = \frac{B_{-1}}{\omega} + \sum_{n=0}^{\infty} B_n\omega^n \tag{41}$$

respectively. The expressions resulting from the terms other than A_{-1} and B_{-1} were previously obtained. Only the integrals

$$I_3 = \int_{-\infty}^{0} \frac{1}{\omega} e^{(\eta\beta_L^*-i\xi)\omega} d\omega + \int_{0}^{\infty} \frac{1}{\omega} e^{-(\eta\beta_L^*+i\xi)\omega} d\omega \tag{42}$$

and

$$I_4 = \int_{-\infty}^{0} \frac{1}{\omega}\left(e^{\eta\beta_L^*\omega} - e^{\eta\beta_T^*\omega}\right)e^{-i\xi\omega} d\omega - \int_{0}^{\infty} \frac{1}{\omega}\left(e^{-\eta\beta_L^*\omega} - e^{-\eta\beta_T^*\omega}\right)e^{-i\xi\omega} d\omega \tag{43}$$

need be considered.

By combining the two integrals in I_3 one finds

$$I_3 = 2i \int_{0}^{\infty} \frac{e^{-\eta\beta_L^*\omega} \sin \xi\omega}{\omega} d\omega = 2i \tan^{-1} \frac{\xi}{\eta\beta_L^*} \; . \tag{44}$$

In a similar manner one finds

$$I_4 = -2 \int_{0}^{\infty} \left(e^{-\eta\beta_L^*\omega} - e^{-\eta\beta_T^*\omega}\right)\frac{\cos \xi\omega}{\omega} d\omega = -\log \frac{\xi^2 + \eta^2\beta_T^{*2}}{\xi^2 + \eta^2\beta_L^{*2}} \; . \tag{45}$$

Substitution of the appropriate coefficients A_n or B_n into Eqs. (35), (37), (44) and (45) leads to expressions for the stresses and accelerations. Retaining only the leading non-vanishing coefficient, A_n or B_n, respectively, one obtains for $V = c_R^*$

$$\sigma_{xx} \sim -\frac{(2-M_T^{*2})}{2\pi VT\Delta_R^*}\left[(2+M_T^{*2}-2M_L^{*2})\tan^{-1}\frac{x}{\beta_L^* z} - (2-M_T^{*2})\tan^{-1}\frac{x}{\beta_T^* z}\right]$$

$$\sigma_{yy} \sim -\frac{(M_T^{*2}-2M_L^{*2})(2-M_T^{*2})}{2\pi VT\Delta_R^*}\tan^{-1}\frac{x}{\beta_L^* z}$$

$$\sigma_{zz} \sim \frac{(2-M_T^{*2})^2}{2\pi VT\Delta_R^*}\left[\tan^{-1}\frac{x}{\beta_L^* z} - \tan^{-1}\frac{x}{\beta_T^* z}\right]$$

$$\sigma_{xz} \sim \frac{\beta_L^*(2-M_T^{*2})}{2\pi VT\Delta_R^*}\log\frac{x^2+\beta_T^{*2}z^2}{x^2+\beta_L^{*2}z^2} \tag{46}$$

$$\ddot{u} \sim \frac{\beta_L^* V}{2\pi m\mu T\Delta_R^*}\left[(2-M_T^{*2})\frac{z}{x^2+\beta_L^{*2}z^2} - 2\beta_T^{*2}\frac{z}{x^2+\beta_T^{*2}z^2}\right]$$

$$\ddot{w} \sim \frac{-\beta_L^* V}{2\pi m\mu T\Delta_R^*}\left[(2-M_T^{*2})\frac{x}{x^2+\beta_L^{*2}z^2} - 2\frac{x}{x^2+\beta_T^{*2}z^2}\right]$$

where

$$M_L^{*2} = \frac{M_L^2}{n}, \quad M_T^{*2} = \frac{M_T^2}{m}, \quad \beta_L^{*2} = 1 - M_L^{*2}, \quad \beta_T^{*2} = 1 - M_T^{*2},$$

and

$$\Delta_R^* = M_T^{*2}\frac{1-m}{m}(2-M_T^{*2}) - \beta_L^*\beta_T^*\left[M_L^{*2}\frac{1-n}{n\beta_L^{*2}} + M_T^{*2}\frac{1-m}{m\beta_T^{*2}}\right].$$

The result (46) differs from Eqs. (39) in two major respects. In the latter, stress and acceleration decay as $\frac{1}{r}$ and $\frac{1}{r^2}$ respectively, where $r^2 = x^2 + z^2$, while if $V = c_R^*$, the stresses do not decay at all with respect to r and the accelerations decay as $\frac{1}{r}$. The situation is somewhat comparable to the resonance of a damped oscillator.

The second significant difference between Eqs. (46) and (39) concerns a matter of symmetry. Quantities σ_{ij} and $\ddot{u}_i$ which were, respectively, symmetric and antisymmetric with respect to x when $V \neq c_R^*$, Eq. (39), are antisymmetric and symmetric, respectively, when $V = c_R^*$.

It is noted that there is no sudden change in behavior concerning decay and symmetry depending on whether $V = c_R^*$ or $V \neq c_R^*$, but there is a gradual transition. This transition in the vicinity of $V = c_R^*$ is not contained in the present results. It would require retention of more than the first significant term in the asymptotic expansion.

Case b

In this case the exponent of the irrotational potential $\overline{\Phi}$ is identical with that for Case a. Therefore, Eqs. (35) and (37) are used again to obtain the irrotational contributions to the stresses and accelerations. The exponent of the equivoluminal potential $\overline{\Psi}$ is, however,

$$h_-(\omega) \equiv h_+(\omega) \equiv h(\omega) = \overline{\beta}_T - i\,\frac{\xi}{\eta} =$$

$$= i\left(m_T^* - \frac{\xi}{\eta}\right) - \omega\,\frac{M_T^{*2}(1-m)}{2mm_T^*} + O(\omega)^2. \quad (47)$$

Since the first term of this expression is purely imaginary, the second term must be retained in order to employ Laplace's method. In Eq. (32) let

$$I = I_5 + I_6 = \int_{-\infty}^{0} g_-(\omega)e^{\eta\omega h(\omega)}\,d\omega + \int_{0}^{\infty} g_+(\omega)e^{\eta\omega h(\omega)}\,d\omega \quad (48)$$

and one finds

$$I_5 \sim \exp\left[-\frac{(\eta m_T^* - \xi)^2}{2\eta M_T^{*2}\,\frac{1-m}{mm_T^*}}\right]\times$$

$$\times \int_0^\infty \sum_{n=0}^{\infty} A_n(-\omega)^n \exp\left[-\omega\sqrt{\eta M_T^{*2}\,\frac{1-m}{2mm_T^*}} + u\,\frac{(\eta m_T^* - \xi)}{\sqrt{2\eta M_T^{*2}\,\frac{1-m}{mm_T^*}}}\right]^2 d\omega \quad (49)$$

where the A_n are defined in Eq. (36).

Retaining only the first non-vanishing coefficient A_n one finds

$$I_5 \sim \frac{A_0 \exp\left[-\frac{(\eta m_T^* - \xi)^2}{2\eta M_T^{*2}\,\frac{1-m}{mm_T^*}}\right]}{\sqrt{\eta M_T^{*2}\,\frac{1-m}{2mm_T^*}}}\,\mathrm{Erfc}\,\frac{i(\eta m_T^* - \xi)}{\sqrt{2\eta M_T^{*2}\,\frac{1-m}{mm_T^*}}} \quad (50)$$

provided $A_0 \neq 0$, while

$$I_5 \sim A_1 \left\{-\frac{1}{\eta M_T^{*2}\,\frac{1-m}{mm_T^*}} + \right.$$

$$\left. + \frac{i(\eta m_T^* - \xi)}{2\left[\eta M_T^{*2}\,\frac{1-m}{2mm_T^*}\right]^{\frac{3}{2}}} \exp\left[-\frac{(\eta m_T^* - \xi)^2}{2\eta M_T^{*2}\,\frac{1-m}{mm_T^*}}\right] \mathrm{Erfc}\,\frac{i(\eta m_T^* - \xi)}{\sqrt{2\eta M_T^{*2}\,\frac{1-m}{mm_T^*}}}\right\} \quad (51)$$

if $A_0 = 0$, $A_1 \neq 0$.

In a similar manner one finds

$$I_6 \sim \frac{B_o \exp\left[-\dfrac{(\eta m_T^* - \xi)^2}{2\eta M_T^{*2}\,\dfrac{1-m}{mm_T^*}}\right]}{\sqrt{\eta M_T^{*2}\,\dfrac{1-m}{2mm_T^*}}}\ \mathrm{Erfc} - \frac{i(\eta m_T^* - \xi)}{\sqrt{2\eta M_T^{*2}\,\dfrac{1-m}{mm_T^*}}} \tag{52}$$

provided $B_o \neq 0$, while

$$I_6 \sim B_1 \left\{ \frac{1}{\eta M_T^{*2}\,\dfrac{1-m}{mm_T^*}} + \right. \tag{53}$$

$$\left. + \frac{i(\eta m_T^* - \xi)}{2\left[\eta M_T^{*2}\,\dfrac{1-m}{2mm_T^*}\right]^{\frac{3}{2}}} \exp\left[-\frac{(\eta m_T^* - \xi)^2}{2\eta M_T^{*2}\,\dfrac{1-m}{mm_T^*}}\right] \mathrm{Erfc} - \frac{i(\eta m_T^* - \xi)}{\sqrt{2\eta M_T^{*2}\,\dfrac{1-m}{mm_T^*}}} \right\}$$

if $B_o = 0$, $B_1 \neq 0$.

Substitution of the appropriate expansion coefficients A_n and B_n into Eqs. (35), (37), (50), (51), (52) and (53) leads to expressions for the stresses and accelerations. The leading terms of these expressions (except for σ_{yy}) are of the type

$$\frac{c}{\sqrt{z}}\, e^{-q\frac{(x-m_T^*z)^2}{z}} \quad \text{or} \quad \frac{c(x-m_T^*z)}{z^{\frac{3}{2}}}\, e^{-q\frac{(x-m_T^*z)^2}{z}}\ \mathrm{Erfc}\left[\frac{i\sqrt{q}(x-m_T^*z)}{\sqrt{z}}\right]$$

where q is a positive constant. All other terms contain, with or without the exponential factor, higher negative powers of z such that the response is only of consequence for ratios $\frac{x}{z}$ near m_T^*. To obtain this significant part of the response it is therefore sufficient to use expressions valid in the vicinity of $\frac{x}{z} = m_T^*$. At other locations the stresses and accelerations may be deemed to vanish in first approximation. Thus, retaining only the leading non-vanishing coefficient, A_n or B_n, respectively, one finds

$$\sigma_{xx} \sim \frac{2m_T^*\beta_L^*}{\pi\Delta_-^*\Delta_+^*} \frac{\exp\left[-\frac{(x - m_T^*z)^2}{2zVTM_T^{*2}\frac{1-m}{mm_T^*}}\right]}{\sqrt{zVTM_T^{*2}\frac{1-m}{2mm_T^*}}} \left[4m_T^*\beta_L^*\sqrt{\pi} - (2-M_T^{*2})^2 i\mathrm{Erf}\frac{i(x - m_T^*z)}{\sqrt{2zVTM_T^{*2}\frac{1-m}{mm_T^*}}}\right]$$

$$\sigma_{yy} \sim \frac{2\beta_L^*(M_T^{*2} - 2M_L^{*2})(2 - M_T^{*2})}{\pi\Delta_-^*\Delta_+^*} \left[\frac{(2 - M_T^{*2})^2 z - 2m_T^*\beta_L^* x}{x^2 + \beta_L^{*2}z^2}\right]$$

$$\sigma_{zz} \sim -\sigma_{xx} \tag{54}$$

$$\sigma_{xz} \sim \frac{\beta_L^*(2-M_T^{*2})}{\pi\Delta_-^*\Delta_+^*} \frac{\exp\left[-\frac{(x - m_T^*z)^2}{2zVTM_T^{*2}\frac{1-m}{mm_T^*}}\right]}{\sqrt{zVTM_T^{*2}\frac{1-m}{2mm_T^*}}} \left[4m_T^*\beta_L^*\sqrt{\pi} - (2-M_T^*)^2 i\mathrm{Erf}\frac{i(x - m_T^*z)}{\sqrt{2zVTM_T^{*2}\frac{1-m}{mm_T^*}}}\right]$$

$$\ddot{u} \sim -\frac{\beta_L^* m_T^*}{\pi m\mu T^2\Delta_-^*\Delta_+^*} \frac{(x - m_T^*z)\sqrt{VT}\exp\left[-\frac{(x - m_T^*z)^2}{2zVTM_T^{*2}\frac{1-m}{mm_T^*}}\right]}{\left[zM_T^{*2}\frac{1-m}{2mm_T^*}\right]^{\frac{3}{2}}} \times$$

$$\times \left[2m_T^*\beta_L^*\sqrt{\pi} + (2 - M_T^{*2})^2 i\mathrm{Erf}\frac{i(x - m_T^*z}{\sqrt{2zVTM_T^{*2}\frac{1-m}{mm_T^*}}}\right]$$

$$w \sim \frac{1}{m_T^*}\ddot{u}$$

where

$$M_L^{*2} = \frac{M_L^2}{n} , \; M_T^{*2} = \frac{M_T^2}{m} , \; \beta_L^{*2} = 1 - M_L^{*2} , \; m_T^{*2} = M_T^{*2} - 1 ,$$

$$\Delta_-^* = (2 - M_T^{*2})^2 - i4m_T^*\beta_L^* , \; \Delta_+^* = (2 - M_T^{*2})^2 + i4m_T^*\beta_L^* ,$$

and the contributions of the irrotational potential Φ have been neglected in all expressions except those for σ_{yy} .

Case c

In this case, since $h_-(\omega) = h_+(\omega)$ and $g_-(\omega) = g_+(\omega)$, Eq. (32) may be written in the form

$$I = \int_{-\infty}^{\infty} g(\omega) e^{\eta\omega h(\omega)} d\omega . \tag{55}$$

Since the exponents are of the form of Eq. (47), the second terms of their expansions must be retained. Thus

$$I \sim \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} A_n \omega^n \exp\left[i\omega(\eta m_T^* - \xi) - \omega^2 \eta M_T^{*2} \frac{1 - m}{2mm_T^*}\right] d\omega \tag{56}$$

where the A_n are defined in Eq. (36).

If the order of summation and integration are interchanged, each term is recognized as a Fourier transform whose inverse may be taken from the tables [13]. Therefore, one finds for the nth term

$$I_n = \frac{\sqrt{\pi}\, A_n}{i^n \sqrt{\eta M_T^{*2} \frac{1 - m}{2mm_T^*}}} \frac{\partial^n}{\partial \xi^n} \exp\left[- \frac{(\eta m_T^* - \xi)^2}{2\eta M_T^{*2} \frac{1 - m}{mm_T^*}}\right] . \tag{57}$$

Substitution of the appropriate expansion coefficients A_n into Eq. (57) leads to expressions for the stresses and accelerations. The leading term in these expressions is of the type

$$\frac{c}{\sqrt{z}}\, e^{-q \frac{(x - m^* z)^2}{z}} \quad \text{or} \quad c\, \frac{(x - m^* x)}{z^{\frac{3}{2}}}\, e^{-q \frac{(x - m^* z)^2}{z}}$$

where q is a positive constant. All other terms contain higher negative powers of z and an exponential factor of the above type such that the response is only of consequence for ratios $\frac{x}{z}$ near m_L^* or m_T^*. To obtain the significant part of the response it is therefore sufficient to retain only the first non-vanishing coefficient A_n. At other locations the response may be deemed to vanish in first approximation. The expressions for the stresses and accelerations thus found for $\frac{x}{z} \simeq m_L^*$ are

$$\begin{aligned}
\sigma_{xx} &\sim \frac{(2 + M_T^{*2} - 2M_L^{*2})(2 - M_T^{*2})}{2\Delta^* \sqrt{z\pi V T M_L^{*2} \frac{1 - m}{2nm_L^*}}} \exp\left[- \frac{(x - m_L^* z)^2}{2zVTM_L^{*2} \frac{1 - n}{nm_L^*}}\right] \\
\sigma_{yy} &\sim \frac{(M_T^{*2} - 2M_L^{*2})}{(2 + M_T^{*2} - 2M_L^{*2})}\, \sigma_{xx} \\
\sigma_{zz} &\sim - \frac{(2 - M_T^{*2})}{(2 + M_T^{*2} - 2M_L^{*2})}\, \sigma_{xx} \qquad (58) \\
\sigma_{xz} &\sim - \frac{2m_L^*}{(2 + M_T^{*2} - 2M_L^{*2})}\, \sigma_{xx}
\end{aligned}$$

$$\ddot{u} \sim -\frac{(2 - M_T^{*2})\sqrt{VT}}{4m\mu T^2\Delta^*\sqrt{\pi}} \frac{(x - m_L^* z)}{\left[zM_L^{*2}\frac{1 - n}{2nm_L^*}\right]^{\frac{3}{2}}} \exp\left[-\frac{(x - m_L^* z)^2}{2zVTM_L^{*2}\frac{1 - n}{nm_L^*}}\right]$$

$$\ddot{w} \sim m_L^* \ddot{u} \qquad (58)$$

cont'd.

and for $\frac{x}{z} \simeq m_T^*$

$$\sigma_{xx} \sim \frac{2m_T^* m_L^*}{\Delta^* \sqrt{z\pi VTM_T^{*2}\frac{1 - m}{2mm_T^*}}} \exp\left[-\frac{(x - m_T^* z)^2}{2zVTM_T^{*2}\frac{1 - m}{mm_T^*}}\right]$$

$$\sigma_{yy} \sim 0$$

$$\sigma_{zz} \sim -\sigma_{xx}$$

$$\sigma_{xz} \sim \frac{(2 - M_T^{*2})}{2m_T^*}\sigma_{xx}$$

$$\ddot{u} \sim -\frac{m_T^* m_L^* \sqrt{VT}}{2m\mu T^2\Delta^*\sqrt{\pi}} \frac{(x - m_T^* z)}{\left[zM_T^{*2}\frac{1 - m}{2mm_T^*}\right]^{\frac{3}{2}}} \exp\left[-\frac{(x - m_T^* z)^2}{2zVTM_T^{*2}\frac{1 - m}{mm_T^*}}\right]$$

$$\ddot{w} \sim \frac{1}{m_T^*}\ddot{u} \qquad (59)$$

where

$$M_L^{*2} = \frac{M_L^2}{n}, \quad M_T^{*2} = \frac{M_T^2}{m}, \quad m_L^{*2} = M_L^{*2} - 1, \quad m_T^{*2} = M_T^{*2} - 1,$$

and

$$\Delta^* = (2 - M_T^{*2})^2 + 4m_L^* m_T^* .$$

V. Asymptotic Approximations for Small Values of η and ξ

For small values of η and ξ the asymptotic evaluation gives in first approximation the rather obvious result that the stresses are equal to those in an elastic medium having the properties of the unrelaxed viscoelastic material. Therefore, no further analysis will be required except for the special situation $V = c_R$, the Rayleigh velocity defined by $\Delta = (2-M_T^2)^2 - 4\beta_L\beta_T = 0.$

In the special situation $V = c_R$ the functions $h(\omega)$ and $g(\omega)\exp[\eta\omega(h(\omega) - h(0))]$ are expanded in powers of $\frac{1}{\omega}$, with the result:

$$h_-(\omega) = \beta_T - i\frac{\xi}{\eta} + O(\tfrac{1}{\omega})$$

$$h_+(\omega) = -\beta_T - i\frac{\xi}{\eta} + O(\tfrac{1}{\omega})$$

$$g_-(\omega)\, e^{\eta\omega[h_-(\omega)-h_-(0)]} = A_2\omega^2 + A_1\omega + A_0 + \sum_{n=1}^{\infty} A_{-n}\omega^{-n} \tag{60}$$

$$g_+(\omega)\, e^{\eta\omega[h_+(\omega)-h_+(0)]} = B_2\omega^2 + B_1\omega + B_0 + \sum_{n=1}^{\infty} B_{-n}\omega^{-n} .$$

The value of the integrals, in this case, is principally due to the contribution from large values of η, such that

$$I = I_7 + I_8 \simeq \int_{-\infty}^{-N} g_-(\omega)e^{\eta\omega h_-(\omega)}\,d\omega + \int_{N}^{\infty} g_+(\omega)e^{\eta\omega h_+(\omega)}\,d\omega \tag{61}$$

where N is a large but finite constant. Using the transformation $\eta\omega = s$ and retaining only the leading non-vanishing coefficient A_n or B_n of the appropriate power series expansion one obtains

$$I_7 \sim \frac{1}{\eta^3} \int_{-\infty}^{-\eta N} A_2 s^2 e^{(\beta_T - i \frac{\xi}{\eta})s} ds \tag{62}$$

provided $A_2 \neq 0$, while

$$I_7 \sim \frac{1}{\eta^2} \int_{-\infty}^{-\eta N} A_1 s \; e^{(\beta_T - i \frac{\xi}{\eta})s} ds \tag{63}$$

if $A_2 = 0$, $A_1 \neq 0$.

For small values of η the limit $-\eta N$ may be replaced by zero. This leads to the result

$$I_7 \sim \frac{2A_2}{(\eta\beta_T - i\xi)^3} \tag{64}$$

if $A_2 \neq 0$, or

$$I_7 \sim - \frac{A_1}{(\eta\beta_T - i\xi)^2} \tag{65}$$

if $A_2 = 0$, $A_1 \neq 0$.

The error introduced by changing the upper limit of integration is NM where M is the maximum value of the function $\underline{g}(\omega) \; \exp(\eta\omega[h_-(\omega)-h_-(0)])$ in the interval $0 > \omega > -N$.

This error can be made as small as desired relative to the value of I_7 by selecting sufficiently small values of ξ and η.

The integral I_8 may be treated in a similar manner with the result

$$I_8 \sim \frac{2B_2}{(\eta\beta_T + i\xi)^3} \tag{66}$$

provided $B_2 \neq 0$, or

$$I_8 \sim \frac{B_1}{(\eta\beta_T + i\xi)^2} \tag{67}$$

if $B_2 = 0,\ B_1 \neq 0$.

Substitution of the appropriate expansion coefficients A_n and B_n into Eqs. (64), (65), (66) and (67) leads to expressions for the stresses and accelerations. Retaining only the leading non-vanishing coefficient one obtains

$$\begin{aligned}
\sigma_{xx} &\sim \frac{VT\beta_L}{\pi\Delta_R}\left[(2+M_T^2-2M_L^2)(2-M_T^2)\frac{xz}{(x^2+\beta_L{}^2z^2)^2}-4\beta_T^2\frac{xz}{(x^2+\beta_T^2z^2)^2}\right] \\
\sigma_{yy} &\sim \frac{VT\beta_L}{\pi\Delta_R}\left[(M_T^2-2M_L^2)(2-M_T^2)\frac{xz}{(x^2+\beta_L^2z^2)^2}\right] \\
\sigma_{zz} &\sim -\frac{VT\beta_L}{\pi\Delta_R}\left[(2-M_T^2)^2\frac{xz}{(x^2+\beta_L^2z^2)^2}-4\beta_T^2\frac{xz}{(x^2\quad\beta_T^2z^2)^2}\right] \\
\sigma_{xz} &\sim -\frac{VT\beta_L(2-M_T^2)}{\pi\Delta_R}\left[\frac{(x^2-\beta_L^2z^2)}{(x^2+\beta_L^2z^2)^2}-\frac{(x^2-\beta_T^2z^2)}{(x^2+\beta_T^2z^2)^2}\right]
\end{aligned} \tag{68}$$

$$\ddot{u} \sim \frac{\beta_L V^3 T}{\pi\mu\Delta_R}\left[(2-M_T^2)\frac{z(\beta_L^2 z^2 - 3x^2)}{(x^2+\beta_L^2 z^2)^3} - 2\beta_T^2 \frac{z(\beta_T^2 z^2 - 3x^2)}{(x^2+\beta_T^2 z^2)^3}\right]$$

(68) cont.

$$\ddot{w} \sim -\frac{\beta_L V^3 T}{\pi\mu\Delta_R}\left[(2-M_T^2)\frac{x(3\beta_L^2 z^2 - x^2)}{(x^2+\beta_L^2 z^2)^3} - 2\frac{x(3\beta_T^2 z^2 - x^2)}{(x^2+\beta_T^2 z^2)^3}\right]$$

where

$$\beta_L^2 = 1 - M_L^2, \; \beta_T^2 = 1 - M_T^2,$$

and

$$\Delta_R = M_T^2(1-m)(2-M_T^2) - \beta_L\beta_T\left[\frac{M_L^2(1-n)}{\beta_L^2} + \frac{M_T^2(1-m)}{\beta_T^2}\right].$$

The situation here is quite similar to the one at large distances when $V = c_R^*$ discussed in connection with Eqs. (46). At small distances $r \to 0$, if $v \neq c_R$ the stress and acceleration increase as $\frac{1}{r}$ and $\frac{1}{r^2}$, respectively, [Ref. 4], where $r^2 = x^2 + z^2$, while if $v = c_R$ the stress and acceleration increase as $\frac{1}{r^2}$ and $\frac{1}{r^3}$, respectively. This is again somewhat comparable to the resonance of a damped oscillator.

The various quantities defined by Eqs. (68) show symmetry or antisymmetry with respect to x opposite to the behavior of the equivalent quantities in the elastic solution for $V \neq c_R$ [Ref. 4].

As in the case $V = c_R^*$, there is no sudden change in behavior concerning decay and symmetry, depending on whether $V = c_R$ or $V \neq c_R$, but there is a gradual transition. This transition in the vicinity of $V = c_R$ is not contained in the result (68). It would require the retention of more than the first significant term in the asymptotic expansion.

VI. Discussion of Results

The asymptotic approximations obtained permit a discussion of the differences between the response for elastic and viscoelastic materials. Deferring discussion of the special situation $V = c_R$, the asymptotic approximations for points at small distances are found to be simply the elastic subsonic solutions, "small" or "large" to be interpreted in comparison with the products of the relaxation time T and the typical wave velocities c_p , c_s , c_p^* and c_s^* . Using the smallest and largest, respectively, the asymptotic approximations for "small" distances apply if

$$r \ll c_s^* T , \tag{69}$$

while the asymptotic approximations for "large" distances apply if

$$r \gg c_p T . \tag{70}$$

The differences between the elastic and viscoelastic situations appear in the far field. These differences depend on the relative values of the velocity V and the wave velocities c_p^* and c_s^* . Three cases, previously designated a , b and c , have been considered.

<u>Case a</u> $\qquad V < c_s^*$

In this case the velocity of the load is less than the propagation velocity of any plane wave in the material, and the effect of viscosity on the stresses and accelerations is not radical. Consider, for example, the stresses as functions of the distance $r = \sqrt{x^2 + z^2}$, but such that $\frac{x}{z}$ is constant, Fig. 6. In the elastic material the stresses are equal to $\frac{c}{r}$, where c depends only on the value of $\frac{x}{z}$. In the viscoelastic material, excluding the special case $V = c_R^*$, the stresses may still be written $\frac{\bar{c}}{r}$, but $\bar{c}$ has a slight dependence on r. For small values of r, $\bar{c} = c$, while for large values of r, $\bar{c}$, Eqs. (39), approaches a slightly different value. As an example, the stress σ_{zz} along $\frac{x}{z} = 0$ is shown in Fig. 7 as a function of z for the specific set of parameters $M_T^2 = 0.3$, $m = 0.5$, $K = \frac{1}{3}$. At large distances the ratio $\frac{\bar{c}}{c}$, in this example, approaches the value 1.56. The value of this ratio depends on the value of $\frac{x}{z}$ and may be negative. Due to the fact that only the asymptotic values have been obtained, the details of the dependence of $\bar{c}$ on r are not known.

The special situation, $V = c_R^*$, may be compared to the response of a damped oscillator at resonance, where stresses and displacements are much larger than at non-resonant frequencies. Similarly, for this value of V the stresses in the far field are much larger than for other values of V. They no longer decay as $\frac{1}{r}$, but approach constant values depending only on the ratio $\frac{x}{z}$. The accelerations which previously decayed as $\frac{1}{r^2}$ now decay as $\frac{1}{r}$. In addition, quantities previously symmetric with respect to the variable x become antisymmetric and vice versa, as discussed in connection with Eq. (46).

Case b $\qquad c_p^* > V > c_s^*$

In this case, where the velocity V exceeds the velocity of shear waves for low frequencies, $\Omega \to 0$, the far field differs radically from the near field, and from the far field in the elastic material. Before proceeding further, the reader is reminded of the character of the supersonic elastic solution, i.e. when $V > c_p > c_s$. In this case the stresses are discontinuous. They vanish everywhere except at points where $\frac{x}{z}$ has one of the values defining the P and S fronts. At these points the stresses become infinite[1]. In the elastic trans-sonic case, the stresses are the sum of a smooth function $\frac{1}{r} c\left(\frac{x}{z}\right)$ and of one which is discontinuous at the S front.

In the viscoelastic case under consideration the response in the near field is again given by the subsonic elastic solution, just as in Case a. The stresses are of the form $\frac{c}{r}$ where c is a smooth function of $\frac{x}{z}$. In the far field, however, the stresses vanish everywhere, in first approximation, except in the vicinity of $\frac{x}{z} = m_T^*$. The presence of the viscous effects changes the response in the far field from a smooth one, into one somewhat similar to the one in the elastic trans-sonic case. In the viscoelastic material large yet finite stresses occur within a small angle in the vicinity of a critical direction defined by $\frac{x}{z} = m_T^*$. In essence, the

[1] While the stress becomes infinite, the integral of the stress across either front is finite and independent of r.

viscoelastic material focuses the response in this critical direction. The focusing is not sharp, but decays exponentially as shown in Eqs. (54). Along the critical direction the stresses, except σ_{yy}, are of the form $\frac{c}{\sqrt{z}}$. As an example, the stress σ_{zz} along $\frac{x}{z} = m_T^*$ is shown in Fig. 8 as a function of z for the specific set of parameters $M_T^2 = 0.75$, $m = 0.5$, $K = \frac{1}{3}$.

<u>Case c</u> $\qquad V > c_p^* > c_s^*$

In this case the far field response in the viscoelastic material again differs radically from that in an elastic material. The situation is somewhat similar to that in Case b, but there is focusing in the two directions $\frac{x}{z} = m_L^*$ and $\frac{x}{z} = m_T^*$. It is to be noted that Case c is only possible if $K > \frac{3}{4 + 3M_T^2 - 4m}$; this can occur only for a very small or negative value of Poisson's ratio in combination with $m \ll 1$.

<u>Special Case $V = c_R$, the Velocity of Rayleigh Waves</u>

In this case the elastic solution does not exist at all. The stresses increase in magnitude as V approaches c_R, just as in the resonance of an undamped oscillator. The presence of viscous effects leads to a finite response, Eqs. (68). The behavior of the solution for small values of r differs radically from the elastic solution for $V \neq c_R$ in two respects. Stresses and accelerations in the latter case increase for $r \to 0$ as $\frac{1}{r}$ and $\frac{1}{r^2}$, respectively, while Eqs. (68) indicate an increase as $\frac{1}{r^2}$ and $\frac{1}{r^3}$, respectively.

In the vicinity of $r = 0$ the response for $V = c_R$ is therefore an order of magnitude larger than for $V \neq c_R$. In addition, there is a change from symmetric to antisymmetric behavior with respect to the variable x and vice versa. The changes are quite similar to those encountered in the far field when $V = c_R^*$.

The far field for $V = c_R$ can be obtained, depending on the material properties, from Case a, b, or c. The fact that $V = c_R$ does not create a special situation in the far field, $r \to \infty$.

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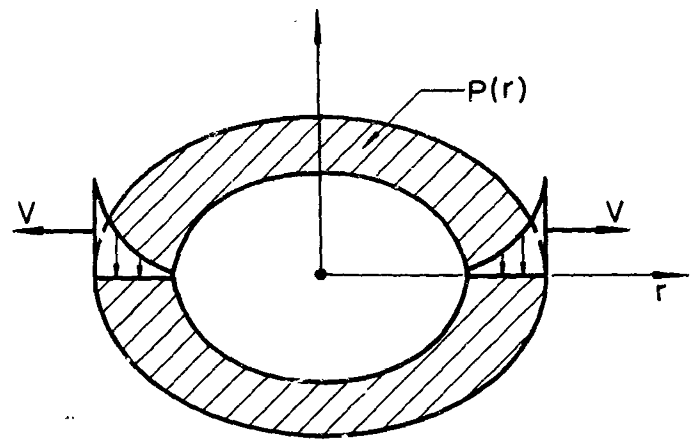
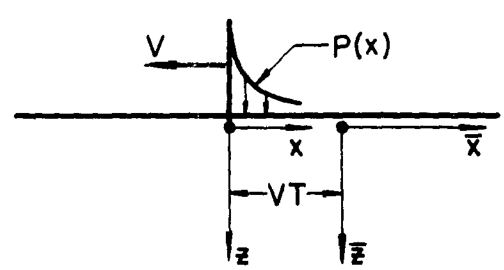


F I G . 1



F I G . 2

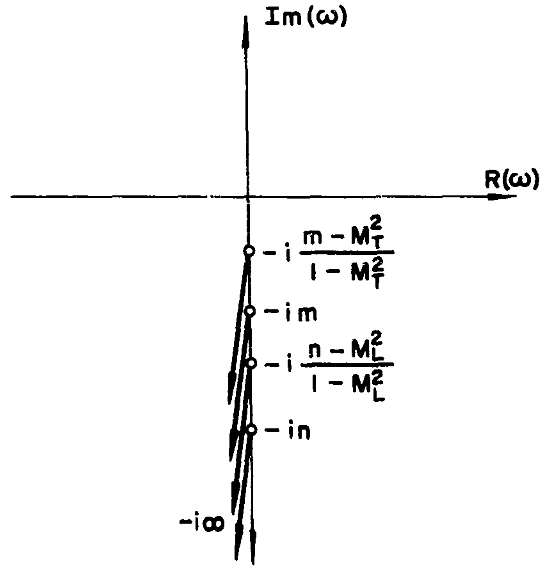


FIG. 3 BRANCH POINTS AND CUTS USED IN CASE a

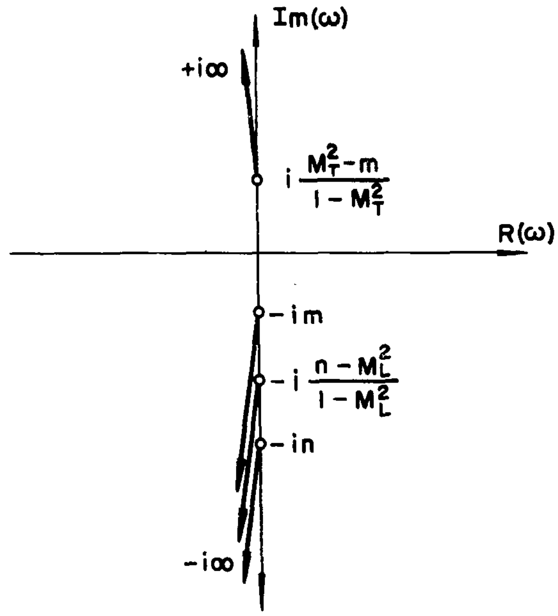


FIG. 4 BRANCH POINTS AND CUTS USED IN CASE b

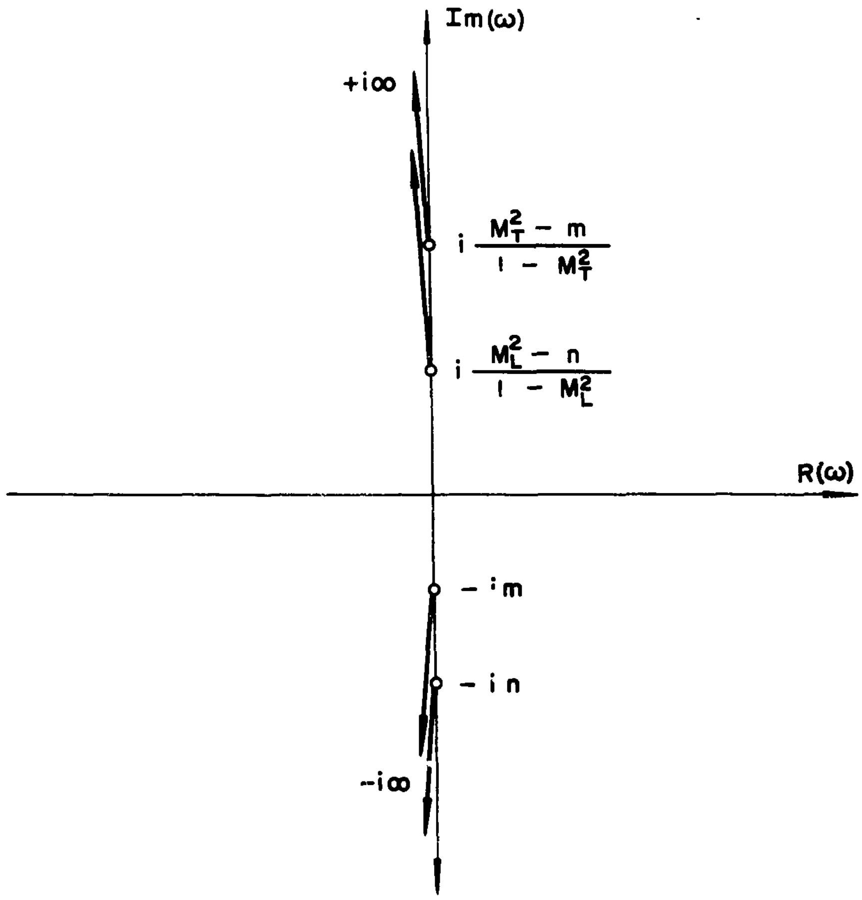


FIG. 5 BRANCH POINTS AND CUTS USED IN CASE c

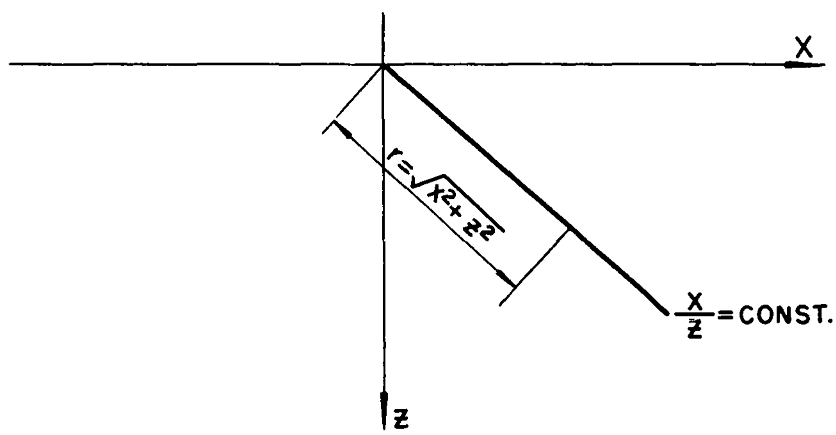
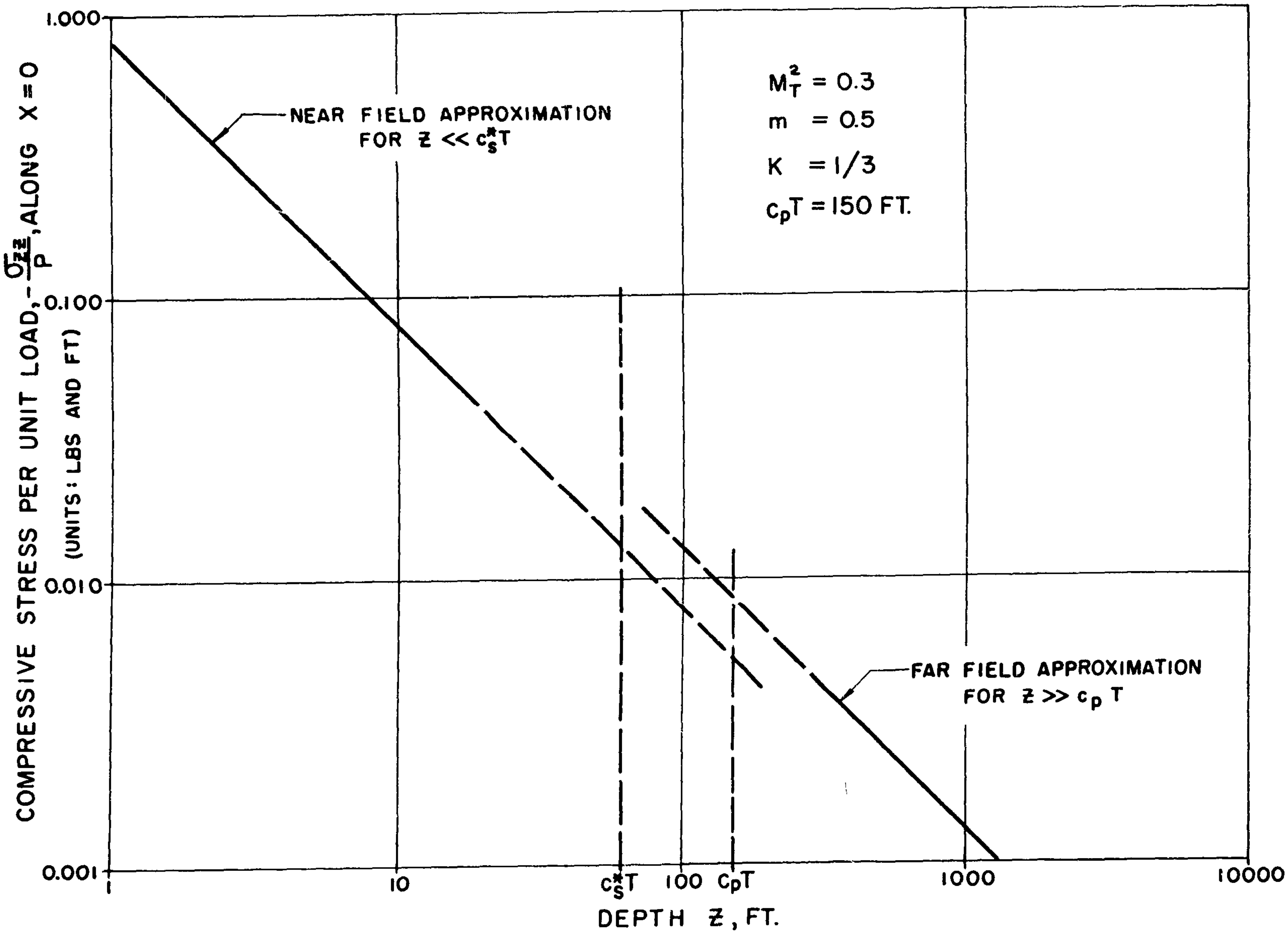
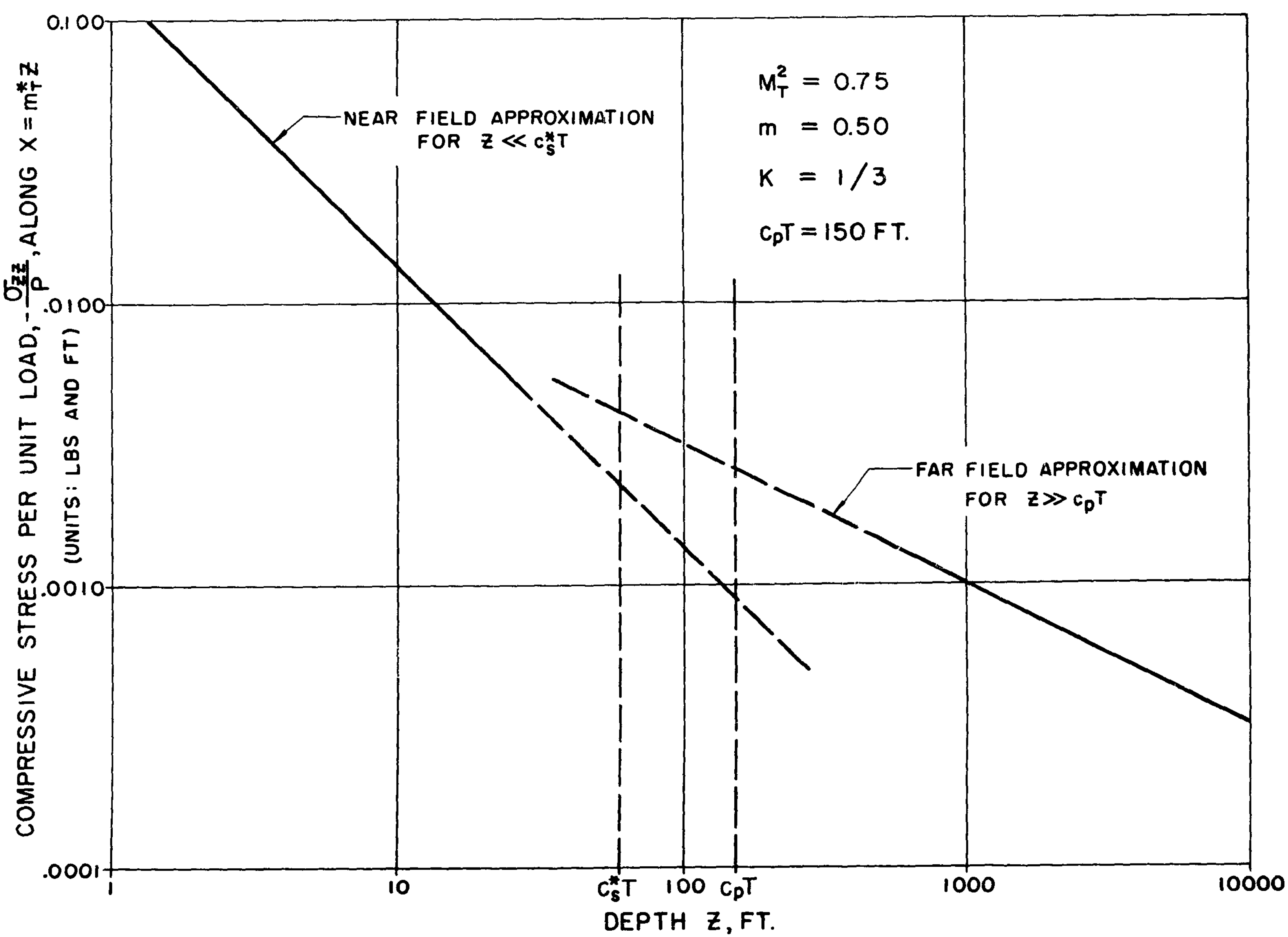


FIG. 6



1.000
0.100
0.010
0.001
COMPRESSIVE STRESS PER UNIT LOAD, $-\frac{\sigma_{zz}}{P}$, ALONG X = 0
(UNITS: LBS AND FT)
NEAR FIELD APPROXIMATION FOR $z << c_s^* T$
$M_T^2 = 0.3$
$m = 0.5$
$K = 1/3$
$c_p T = 150$ FT.
FAR FIELD APPROXIMATION FOR $z >> c_p T$
1
10
$c_s^* T$
100
$c_p T$
1000
10000
DEPTH z, FT.

FIG. 7



0.100
.0100
.0010
.0001
1
10
c_s^*T
100
c_pT
1000
10000
COMPRESSIVE STRESS PER UNIT LOAD, $-\frac{\sigma_{zz}}{P}$, ALONG $X = m_T^* Z$
(UNITS: LBS AND FT)
DEPTH Z, FT.
NEAR FIELD APPROXIMATION
FOR $Z \ll c_s^*T$
FAR FIELD APPROXIMATION
FOR $Z \gg c_pT$
$M_T^2 = 0.75$
$m = 0.50$
$K = 1/3$
$c_pT = 150$ FT.

FIG. 8